Particle in a 2-D Box: Problem TZDII[1] 8.12

Class Work, Wednesday, 1/24/24

Consider a particle in a 2-D box with sides of length a.
The wave function is

$$\psi(x,y) = A\sin\left(\frac{n_x \pi x}{a}\right)\sin\left(\frac{n_y \pi y}{a}\right) \quad \text{TZDII (8.23)}$$

Write down probability density $|\psi|^2$ and determine where the particle is most likely to be found. How many such points are there? Sketch a contour map similar to those in Fig. 8.4. For

a) $n_x = 1$, $n_y = 2$
b) $n_x = 2$, $n_y = 2$
c) $n_x = 4$, $n_y = 3$

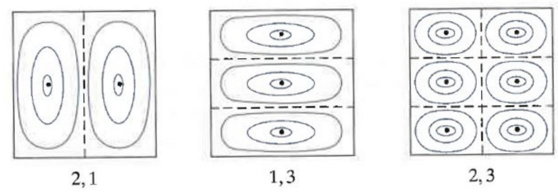


FIGURE 8.4
Contour maps of $|\psi|^2$ for three excited states of the square box. The two numbers under each picture are n_x and n_y. The dashed lines are nodal lines, where $|\psi|^2$ vanishes; these occur where ψ passes through zero as it oscillates from positive to negative values.

a) For $n_x = 1$ and $n_y = 2$, the probability density $|\psi(x,y)|^2$ is $|\psi(x,y)|^2 = A^2\sin^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{2\pi y}{a}\right)$

Find A by normalization

$$\int_0^a\int_0^a |\psi(x,y)|^2 dxdy = A^2\int_0^a\int_0^a \sin^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{2\pi y}{a}\right)dxdy = 1$$

Integral #176 in the CRC gives

$$\int \sin^2(ax)dx = \frac{1}{2}x - \frac{1}{4a}\sin(2ax)$$

Thus, in general

$$\int_0^a\int_0^a |\psi(x,y)|^2 dxdy = A^2\left[\frac{x}{2} - \frac{a}{4n_x\pi}\sin\left(\frac{4n_x\pi x}{a}\right)\Bigg|_0^a\right]\left[\frac{y}{2} - \frac{a}{4n_y\pi}\sin\left(\frac{4n_x\pi y}{a}\right)\Bigg|_0^a\right] = 1$$

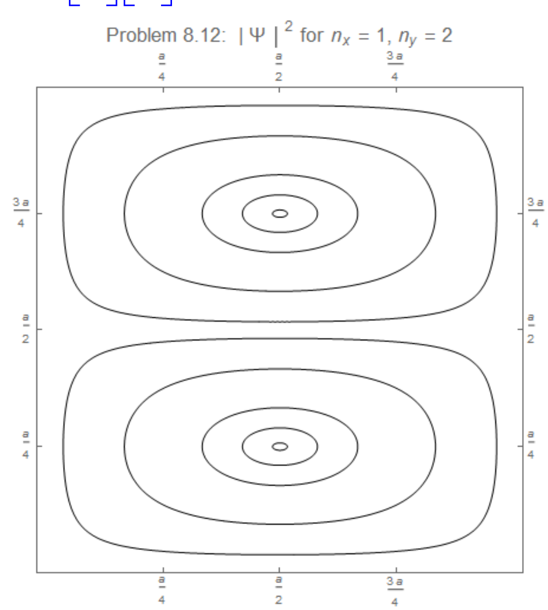
For $n_x = 1$ and $n_y = 2$, this becomes

$$A^2\left[\frac{x}{2} - \frac{a}{4n_x\pi}\sin\left(\frac{4\pi x}{a}\right)\Bigg|_0^a\right]\left[\frac{y}{2} - \frac{a}{4n_y\pi}\sin\left(\frac{8\pi y}{a}\right)\Bigg|_0^a\right] = A^2\left[\frac{a}{2}\right]\left[\frac{a}{2}\right] = 1 \quad \Rightarrow \quad A = \frac{2}{a}$$

And the probability density is

$$|\psi(x,y)|^2 = \frac{4}{a^2}\sin^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{2\pi y}{a}\right)$$

Making a contour plot of this in Mathematica (see T:\O'Donoghue\Modern\TZDIIPr08_12) gives the image to the right. This has one maximum in x at x = 0.5a and two in y at y = 0.25a & y = 0.75a.



[1] Modern Physics for Scientists and Engineers, 2nd Ed., John R. Taylor, Chris D. Zafiratos, & Michael A. Dubson (Prentice Hall, 2002

$$\psi(x,y) = A\sin\left(\frac{n_x \pi x}{a}\right)\sin\left(\frac{n_y \pi y}{a}\right)$$

b) For $n_x = 2$ and $n_y = 2$, the probability density $|\psi(x,y)|^2$, for $n_x = 1$ and $n_y = 2$ is

$$|\psi(x,y)|^2 = A^2 \sin^2\left(\frac{2\pi x}{a}\right)\sin^2\left(\frac{2\pi y}{a}\right)$$

Find A by normalization

$$\int_0^a \int_0^a |\psi(x,y)|^2 dxdy = A^2 \int_0^a \int_0^a \sin^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{2\pi y}{a}\right)dxdy = 1$$

Integral #176 in the CRC gives

$$\int \sin^2(ax)dx = \frac{1}{2}x - \frac{1}{4a}\sin(2ax)$$

Thus, in general

$$\int_0^a \int_0^a |\psi(x,y)|^2 dxdy = A^2\left[\frac{x}{2} - \frac{a}{4n_x\pi}\sin\left(\frac{4n_x\pi x}{a}\right)\Bigg|_0^a\right]\left[\frac{y}{2} - \frac{a}{4n_y\pi}\sin\left(\frac{4n_x\pi y}{a}\right)\Bigg|_0^a\right] = 1$$

For $n_x = 1$ and $n_y = 2$, this becomes

$$A^2\left[\frac{x}{2} - \frac{a}{4n_x\pi}\sin\left(\frac{8\pi x}{a}\right)\Bigg|_0^a\right]\left[\frac{y}{2} - \frac{a}{4n_y\pi}\sin\left(\frac{8\pi y}{a}\right)\Bigg|_0^a\right] = A^2\left[\frac{a}{2}\right]\left[\frac{a}{2}\right] = 1 \quad \Rightarrow \quad A = \frac{2}{a}$$

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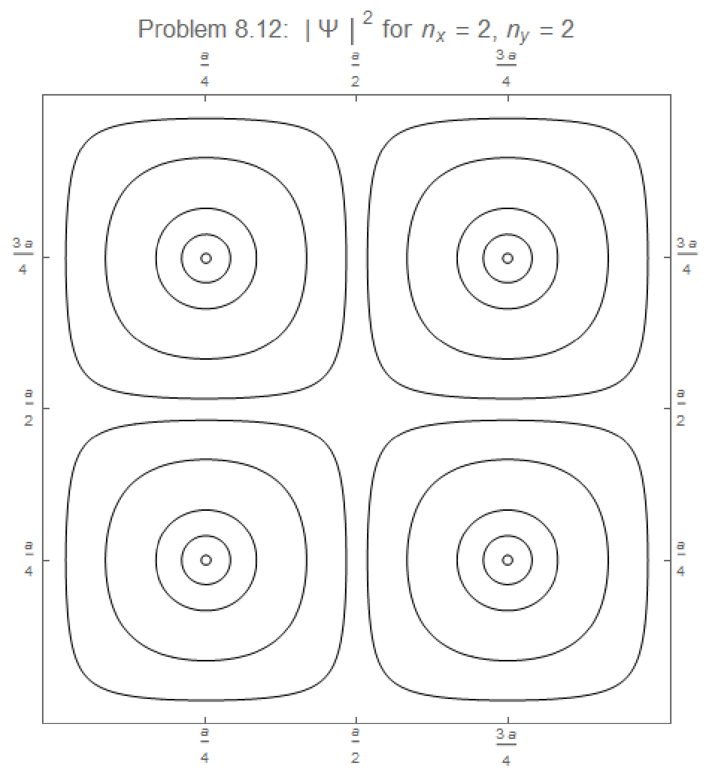
Making a contour plot of this in Mathematica (see T:\O'Donoghue\Modern\TZDIIPr08_12) gives the image to the right. This has two maxima in each direction at x = 0.25a & x = 0.75a and y = 0.25a & y = 0.75a.



$$\psi(x,y) = A\sin\left(\frac{n_x \pi x}{a}\right)\sin\left(\frac{n_y \pi y}{a}\right)$$

c) For $n_x = 4$ and $n_y = 3$, the probability density $|\psi(x,y)|^2$ is

$$|\psi(x,y)|^2 = A^2 \sin^2\left(\frac{4\pi x}{a}\right)\sin^2\left(\frac{3\pi y}{a}\right)$$

Find A by normalization

$$\int_0^a \int_0^a |\psi(x,y)|^2 dxdy = A^2 \int_0^a \int_0^a \sin^2\left(\frac{4\pi x}{a}\right)\sin^2\left(\frac{3\pi y}{a}\right)dxdy = 1$$

Integral #176 in the CRC gives

$$\int \sin^2(ax)dx = \frac{1}{2}x - \frac{1}{4a}\sin(2ax)$$

Thus, in general

$$\int_0^a \int_0^a |\psi(x,y)|^2 dxdy = A^2\left[\frac{x}{2} - \frac{a}{4n_x\pi}\sin\left(\frac{4n_x \pi x}{a}\right)\Bigg|_0^a\right]\left[\frac{y}{2} - \frac{a}{4n_y\pi}\sin\left(\frac{4n_x \pi y}{a}\right)\Bigg|_0^a\right] = 1$$

For $n_x = 1$ and $n_y = 2$, this becomes

$$A^2\left[\frac{x}{2} - \frac{a}{4n_x\pi}\sin\left(\frac{8\pi x}{a}\right)\Bigg|_0^a\right]\left[\frac{y}{2} - \frac{a}{4n_y\pi}\sin\left(\frac{6\pi y}{a}\right)\Bigg|_0^a\right] = A^2\left[\frac{a}{2}\right]\left[\frac{a}{2}\right] = 1 \quad \Rightarrow \quad A = \frac{2}{a}$$

And the probability density is

$$|\psi(x,y)|^2 = \frac{4}{a^2}\sin^2\left(\frac{4\pi x}{a}\right)\sin^2\left(\frac{3\pi y}{a}\right)$$

Making a contour plot of this in Mathematica (see T:\O'Donoghue\Modern\TZDIIPr08_12) gives the image to the right. This has four maxima in x at x = a/8, 3a/8, 5a/8, & 7a/8 and three maxima in y at y = a/6, 3a/6, & 5a/6.